The nonstandard constrained KP hierarchy and the generalized Miura transformations

Ming-Hsien Tu
Department of Physics, National Tsing Hua University,
Hsinchu, Taiwan, Republic of China.
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Abstract

We consider the nonstandard constrained KP (ncKP) hierarchy which is obtained from the multi-constraint KP hierarchy by gauge transformation. The second Hamiltonian structure of the ncKP hierarchy can be simplified by factorizing the Lax operator into multiplication form, thus the generalized Miura transformation is obtained. We also discuss the free field realization of the associated W-algebra.

I. INTRODUCTION

In the past few years, there are several intensive studies on the relationships between conformal field theory and integrable system which include, in particular, exploration of the role played by the classical W-algebras in integrable systems [1]. It's Adler map (see, for example, [2]) from which the W-algebras can be constructed as Poisson bracket algebras. A typical example is the W_n algebra constructed from the second Gelfand-Dickey (GD) structure of the n-th Korteweg-de Vries (KdV) hierarchy [3,4]. Amazingly, by factorization of the KdV-Lax operator, the second Hamiltonian structure is transformed into a much simpler one in an appropriate space of the modified variables. Thus the factorization not only provides a Miura transformation which maps the n-th KdV hierarchy to the corresponding modified hierarchies, but also gives a free field realization of the W_n algebra. This is what we called the Kupershmidt-Wilson (KW) theorem [5,6]. The generalization of the KW theorem to the Kadomtsev-Petviashvili (KP) hierarchy and its reductions have been discussed [7–13]. In general, the above scheme is encoded in the particular form of the Lax operator and its associated Poisson structure. Therefore, the number of integrable hierarchies where the KW theorem works is quite limited.

Recently, Q P Liu [14] conjectured that the above scheme also works for the constrained modified KP (cmKP) hierarchy [15], which is a kind of reduction of the KP hierarchy. The proof has been given in two recent papers [16,17] based on the observation [18] that the second Hamiltonian structure of the cmKP hierarchy can be mapped into the sum of the second and the third GD brackets. Therefore one can factorize the Lax operator of the cmKP hierarchy into linear terms. In this paper, we generalize the previous results [16,17] to the nonstandard constrained KP (ncKP) hierarchy [15], which is obtained from the gauge transformation of the multi-constraint KP hierarchy [15]. We find that the second Poisson structure of the ncKP hierarchy can be simplified by factorizing the nonstandard Lax operator into multiplication form containing inverse linear terms.

This paper is organized as follows: In Sec.II we consider the multi-constraint KP hierarchy. Using the 2-constraint KP hierarchy as an example, we calculate its Poisson brackets from its second Hamiltonian structure and discuss its associated conformal algebra. Then in Sec. III we perform a gauge transformation to obtained the nonstandard cKP hierarchy and the corresponding Poisson brackets. We find that after mapping the nonstandard Lax operator to a 1-constraint KP Lax operator, the Poisson structure becomes the sum of the second and the third GD brackets defined by the 1-constraint KP Lax operator. We also show that the conformal algebra associated with the nonstandard Lax operator is encoded in the conformal algebra of the 1-constraint KP Lax operator. In Sec. IV we simplify this Poisson structure by factorizing the Lax operator into multiplication form and thus obtain the generalized Miura transformation. Conclusions and discussions are presented in Sec. V.

II. MULTI-CONSTRAINT KP HIERARCHY

The multi-constraint KP hierarchy is the ordinary KP hierarchy restricted to pseudo-differential operator of the form

$$L_{(N,M)} = \partial^N + u_2\partial^{N-2} + \cdots + u_N + \sum_{i=1}^{M} \phi_i \partial^{-1} \psi_i. \tag{2.1}$$

The evolution of the system is given by

$$\partial_k L_{(N,M)} = [(L^{k/N}_{(N,M)})_+, L_{(N,M)}], \tag{2.2}$$

$$\partial_k \phi_i = ((L^{k/N}_{(N,M)})_+ \phi_i)_0, \qquad \partial_k \psi_i = -((L^{k/N}_{(N,M)})^*_+ \psi_i)_0 \tag{2.3}$$

where ϕ_i and ψ_i are eigenfunctions and adjoint eigenfunctions, respectively. (Notations:$(A)_\pm$ denote the differential part and the integral part of the pseudo-differential operator A respectively, $(A)_0$ denotes the zeroth order term, and * stands for the conjugate operation: $(AB)^* = B^* A^*,\ \partial^* = -\partial,\ f(x)^* = f(x)$).

The second Hamiltonian structure associated with $L_{(N,M)}$ is given by the second GD bracket as follow

$$\Theta_2^{GD}(\frac{\delta H}{\delta L_{(N,M)}}) = (L_{(N,M)} \frac{\delta H}{\delta L_{(N,M)}})_+ L_{(N,M)} - L_{(N,M)} (\frac{\delta H}{\delta L_{(N,M)}} L_{(N,M)})_+ + \frac{1}{N}[L_{(N,M)}, \int^x res[L_{(N,M)}, \frac{\delta H}{\delta L_{(N,M)}}]]. \tag{2.4}$$

where the last term in (2.4) is just the Dirac constraint imposed by $u_1 = 0$ on $L_{(N,M)}$.

In the following, we will discuss the simplest example $(N = 1, M = 2)$ in detail.

The 2-constraint KP hierarchy with order one is defined by

$$L_{(1,2)} = \partial + \phi_1 \partial^{-1} \psi_1 + \phi_2 \partial^{-1} \psi_2 \tag{2.5}$$

From (2.4) the basic second Poisson brackets are given by

$$\begin{aligned} \{\phi_i, \phi_j\} &= -(\phi_i \partial^{-1} \phi_j + \phi_j \partial^{-1} \phi_i), \\ \{\psi_i, \psi_j\} &= -(\psi_i \partial^{-1} \psi_j + \psi_j \partial^{-1} \psi_i), \\ \{\phi_i, \psi_j\} &= (\delta_{ij} L_{(1,2)} + \phi_i \partial^{-1} \psi_j), \end{aligned} \tag{2.6}$$

which is obviously nonlocal. The algebraic structure of the Poisson brackets is transparent if we set $t \equiv \phi_1\psi_1 + \phi_2\psi_2$, then

$$\begin{aligned} \{t, t\} &= 2t\partial + t', \\ \{\phi_i, t\} &= \phi_i \partial + \phi_i' \\ \{\psi_i, t\} &= \psi_i \partial + \psi_i'. \end{aligned} \tag{2.7}$$

Hence ϕ_i and ψ_i are spin-1 fields with respect to the Virasoro generator t, and (2.7) form a nonlocal extension of the Virasoro algebra by four spin-1 fields. We would like to remark that the algebra (2.7) can be generalized to the multi-constraint case $(N = 1, M > 2)$ by setting $t = \sum_{i=1}^{M} \phi_i \psi_i$.

III. NONSTANDARD CKP HIERARCHY

The nonstandard Lax operator is obtained by performing a gauge transformation on $L_{(1,2)}$ as follow

$$K_{(1,2)} = \phi_1^{-1} L_{(1,2)} \phi_1, \tag{3.1}$$
$$= \partial + v_1 + \partial^{-1} v_2 + q\partial^{-1} r \tag{3.2}$$

where

$$v_1 = \phi_1'/\phi_1, \qquad v_2 = \phi_1 \psi_1, \tag{3.3}$$
$$q = \phi_1^{-1}\phi_2, \qquad r = \phi_1 \psi_2. \tag{3.4}$$

The transformed Lax operator $K_{(1,2)}$ satisfies the hierarchy equations

$$\begin{aligned} \partial_n K_{(1,2)} &= [(K_{(1,2)}^n)_{\geq 1}, K_{(1,2)}], \\ \partial_n q &= ((K_{(1,2)}^n)_{\geq 1} q)_0, \\ \partial_n v_2 &= -((K_{(1,2)}^n)^*_{\geq 1} v_2)_0, \qquad \partial_n r = -((K_{(1,2)}^n)^*_{\geq 1} r)_0. \end{aligned} \tag{3.5}$$

and the transformed second Hamiltonian structure now becomes [15]

$$\begin{aligned} \Theta_2^{NS}(\frac{\delta H}{\delta K_{(1,2)}}) = &(K_{(1,2)}\frac{\delta H}{\delta K_{(1,2)}})_+ K_{(1,2)} - K_{(1,2)}(\frac{\delta H}{\delta K_{(1,2)}} K_{(1,2)})_+ + [K_{(1,2)}, (K_{(1,2)}\frac{\delta H}{\delta K_{(1,2)}})] \\ &+ \partial^{-1} res[K_{(1,2)}, \frac{\delta H}{\delta K_{(1,2)}}] K_{(1,2)} + [K_{(1,2)}, \int^x res[K_{(1,2)}, \frac{\delta H}{\delta K_{(1,2)}}]]. \end{aligned} \tag{3.6}$$

where the basic Poisson brackets can be easily written as

$$\begin{aligned} \{v_1, v_1\} &= 2\partial, \\ \{v_1, v_2\} &= \partial^2 + \partial v_1 + \partial q \partial^{-1} r, \\ \{v_1, q\} &= -q' \partial^{-1}, \\ \{v_1, r\} &= -r, \\ \{v_2, v_2\} &= \partial v_2 + v_2 \partial + v_2 q \partial^{-1} r + r\partial^{-1} q v_2, \\ \{v_2, q\} &= -\partial q + v_1 q - v_2 q \partial^{-1} - r\partial^{-1} q^2, \\ \{v_2, r\} &= \partial r - v_1 r + r\partial^{-1} q r - r\partial^{-1} v_2, \\ \{q, q\} &= -2q\partial^{-1} q + \partial^{-1} q^2 + q^2 \partial^{-1}, \\ \{q, r\} &= \partial + v_1 + 2q\partial^{-1} r + \partial^{-1} v_2 - \partial^{-1} q r, \\ \{r, r\} &= -2r\partial^{-1} r. \end{aligned} \tag{3.7}$$

These Poisson brackets are nonlocal as well. To simplified the above Poisson brackets, we may consider the operator

$$L_{(2,1)} = \partial K_{(1,2)} \tag{3.8}$$
$$= \partial^2 + u_1 \partial + u_2 + \phi \partial^{-1} \psi \tag{3.9}$$

where

$$\begin{aligned} u_1 &= v_1, \qquad u_2 = v_2 + v_1' + qr, \\ \phi &= q', \qquad \psi = r. \end{aligned} \tag{3.10}$$

Using (3.7) we can calculate the Poisson brackets for $\{u_1, u_2, \phi, \psi\}$ which now become simpler

$$\begin{aligned} \{u_1, u_1\} &= 2\partial, \\ \{u_1, u_2\} &= -\partial^2 + \partial u_1, \\ \{u_1, \phi\} &= \phi, \\ \{u_1, \psi\} &= -\psi, \\ \{\phi, \phi\} &= -2\phi\partial^{-1}\phi \end{aligned} \tag{3.11}$$

etc. Note that these brackets are not the same as the ones constructed from the second GD brackets for the 1-constraint KP hierarchy where the corresponding brackets are given by [15]

$$\begin{aligned} \{u_1, u_1\} &= -2\partial, \\ \{u_1, u_2\} &= \partial^2 - \partial u_1, \\ \{u_1, \phi\} &= -\phi, \\ \{u_1, \psi\} &= \psi, \\ \{\phi, \phi\} &= -\phi\partial^{-1}\phi. \end{aligned} \tag{3.12}$$

etc. In fact, it can be shown [see Appendix] that (3.11) obey the following Poisson structure

$$\{F, G\} = \int res(\frac{\delta F}{\delta K_{(1,2)}}\Theta_2^{NS}(\frac{\delta G}{\delta K_{(1,2)}})) = \int res(\frac{\delta F}{\delta L_{(2,1)}}\Omega(\frac{\delta G}{\delta L_{(2,1)}})) \tag{3.13}$$

where

$$\Omega(\frac{\delta G}{\delta L_{(2,1)}}) = (L_{(2,1)}\frac{\delta G}{\delta L_{(2,1)}})_+ L_{(2,1)} - L_{(2,1)}(\frac{\delta G}{\delta L_{(2,1)}}L_{(2,1)})_+ + [L_{(2,1)}, \int^x res[L_{(2,1)}, \frac{\delta G}{\delta L_{(2,1)}}]]. \tag{3.14}$$

Besides the second GD structure, the last term in (3.14) is called the third GD structure which is compatible with the second one [3]. Thus the Hamiltonian structure associated with $L_{(2,1)}$ is the sum of the second and the third GD structures.

Before ending this section, let us discuss the algebraic structure associated with the ncKP hierarchy. Based on the dimension consideration, we can define a Virasoro generator $t \equiv v_2 + v_1'/2 + qr$. Then from (3.7), we have

$$\begin{aligned} \{v_1, t\} &= v_1\partial + v_1', \\ \{t, t\} &= \frac{1}{2}\partial^3 + 2t\partial + t', \\ \{q, t\} &= \frac{1}{2}q\partial + q' - \frac{1}{2}\partial^{-1}q\partial^2, \\ \{r, t\} &= \frac{3}{2}r\partial + r'. \end{aligned} \tag{3.15}$$

We see that v_1 and r are spin-1 and spin-3/2 fields, respectively and q is not a spin field due to the anomalus term "$-\frac{1}{2}\partial^{-1}q\partial^2$ ". However, if we take a derivative to the third bracket in (3.15), then q' becomes a spin-3/2 field, i.e.

$$\{q', t\} = \frac{3}{2}q'\partial + q''. \tag{3.16}$$

This motivate us to covariantize the Lax operator $L_{(2,1)}$ rather than the operator $K_{(1,2)}$. Form (3.11), $L_{(2,1)}$ can be covariantized by setting the Virasoro generator $t \equiv u_2 - 1/2u_1'$, and

$$\begin{aligned}
\{u_1, t\} &= u_1\partial + u_1', \\
\{t, t\} &= \frac{1}{2}\partial^3 + 2t\partial + t', \\
\{\phi, t\} &= \frac{3}{2}\phi\partial + \phi', \\
\{\psi, t\} &= \frac{3}{2}\psi\partial + \psi'.
\end{aligned} \tag{3.17}$$

Therefore, the conformal algebra associated with $K_{(1,2)}$ is encoded in the conformal algebra of $L_{(2,1)}$.

IV. THE GENERALIZED MIURA TRANSFORMATION

In this section, we will show that the Poisson structure (3.14) has a very interesting property under factorization of the operator $L_{(2,1)}$ into multiplication form. Since the operator of the form $L_{(2,1)}$ has multi-boson representations, we can factorize $L_{(2,1)}$ into the following form

$$L_{(2,1)} = (\partial - a_1)(\partial - a_2)(\partial - a_3)(\partial - b_1)^{-1} \tag{4.1}$$

where the variables $\{u_1, u_2, \phi, \psi\}$ and $\{a_1, a_2, a_3, b_1\}$ are related by

$$\begin{aligned}
u_1 &= b_1 - (a_1 + a_2 + a_3), \\
u_2 &= u_1 b_1 + 2b_1' + a_1a_2 + a_2a_3 + a_1a_3 - a_2' - 2a_3', \\
\phi &= e^{\int^x b_1}(u_2 b_1 + u_1 b_1' + b_1'' - a_1a_2a_3 + a_1a_3' + a_2'a_3 + a_2a_3' - a_3''), \\
\psi &= e^{-\int^x b_1}
\end{aligned} \tag{4.2}$$

which is called the Miura transformation. Now let us first consider the second GD bracket under the factorization (4.1). Thanks to the generalized KW theorem [10–13], the second GD bracket can be simplified as

$$\begin{aligned}
\{a_i, a_j\}_2^{GD} &= -\delta_{ij}\partial, \\
\{b_1, b_1\}_2^{GD} &= \partial, \\
\{a_i, b_1\}_2^{GD} &= 0,
\end{aligned} \tag{4.3}$$

Hence the remaining tasks are to study the third GD structure. In the previous paper [16], we have shown that the third structure has also a very nice property under factorization of the Lax operator containing inverse linear terms (4.1). It turns out that [16]

$$\{F,G\}_3^{GD} = \int res(\frac{\delta F}{\delta L_{(2,1)}}[L_{(2,1)}, \int^x res[L_{(2,1)}, \frac{\delta G}{\delta L_{(2,1)}}]]) = (\sum_{i=1}^{3}\frac{\delta F}{\delta a_i}+\frac{\delta G}{\delta b_1})(\sum_{j=1}^{3}\frac{\delta F}{\delta a_j}+\frac{\delta G}{\delta b_1})' \tag{4.4}$$

which leads to

$$\{a_i, a_j\}_3^{GD} = \{a_i, b_1\}_3^{GD} = \{b_1, b_1\}_3^{GD} = \partial. \tag{4.5}$$

Combining (4.3) with (4.5) we obtain

$$\begin{aligned} \{a_i, a_j\} &= (1-\delta_{ij})\partial, \\ \{b_1, b_1\} &= 2\partial, \\ \{a_i, b_1\} &= \partial. \end{aligned} \tag{4.6}$$

Therefore, the Lax operator $K_{(1,2)}$ (and hence $L_{(1,2)}$) has a simple and local realization of their Poisson structures.

V. CONCLUSIONS

We have shown that the second Hamiltonian structure of the ncKP hierarchy has a very simple realization. In terms of $\{a_1, a_2, a_3, b_1\}$ the Lax operator $K_{(1,2)}$ can be factorized as

$$K_{(1,2)} = \partial^{-1}(\partial - a_1)(\partial - a_2)(\partial - a_3)(\partial - b_1)^{-1} \tag{5.1}$$

and the second Poisson structure (3.7) is mapped to a much simpler form (4.6). In general, we should consider the multi-constraint KP hierarchy with the Lax operator of the form (2.1). After performing the gauge transformation $K_{(N,M)} = \phi_1^{-1}L_{(N,M)}\phi_1$, the Lax operator $L_{(N,M)}$ is transformed to

$$K_{(N,M)} = \partial^N + v_1\partial^{N-1} + \cdots + v_N + \partial^{-1}v_{N+1} + \sum_{i=1}^{M-1} q_i\partial^{-1}r_i \tag{5.2}$$

which satisfies the nonstandard hierarchy equations (3.5) and has the Hamiltonian structure (3.6). Moreover we can follow the strategy in Appendix to prove without difficulty that the Hamiltonian structure associated with the operator $L_{(N+1,M-1)} \equiv \partial K_{(N,M)}$ is just the sum of the second and third GD structure (3.14). Thus by applying the previous results [16], the Lax operator of the ncKP hierarchy can be factorized as

$$K_{(N,M)} = \partial^{-1}(\partial - a_1)\cdots(\partial - a_n)(\partial - b_1)^{-1}\cdots(\partial - b_m)^{-1} \tag{5.3}$$

and the simplified Poisson brackets turn out to be

$$
\begin{aligned}
\{a_i, a_j\} &= (1-\delta_{ij})\partial, \\
\{b_i, b_j\} &= (1+\delta_{ij})\partial, \\
\{a_i, b_j\} &= \partial.
\end{aligned}
\tag{5.4}
$$

Finally we would like to remark that the Poisson bracket matrix (5.4) is symmetric and nonsingular, thus it is not difficult to diagonalize the matrix to obtain the free field representation which would be useful to quantize the W-algebra associated with the ncKP hierarchy. The details of these discussions will be presented in a forthcoming paper [19].

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APPENDIX A:

In this appendix we give a proof of (3.13). From (3.10) we have

$$
\frac{\delta H}{\delta v_1} = \frac{\delta H}{\delta u_1} - (\frac{\delta H}{\delta u_2})', \qquad \frac{\delta H}{\delta v_2} = \frac{\delta H}{\delta u_2}, \tag{A1}
$$

$$
\frac{\delta H}{\delta q} = r\frac{\delta H}{\delta v_2} - (\frac{\delta H}{\delta \phi})', \qquad \frac{\delta H}{\delta r} = q\frac{\delta H}{\delta v_2} + \frac{\delta H}{\delta \psi}. \tag{A2}
$$

Let

$$
\frac{\delta H}{\delta K_{(1,2)}} = \partial^{-1}\frac{\delta H}{\delta v_1} + \frac{\delta H}{\delta v_2} + A \tag{A3}
$$

where $A = (A)_{\geq 0}$. Then from

$$
\delta H = \int res(\frac{\delta H}{\delta K_{(1,2)}}\delta K_{(1,2)}) = \int (\frac{\delta H}{\delta v_1}\delta v_1 + \frac{\delta H}{\delta v_2}\delta v_2 + \frac{\delta H}{\delta q}\delta q + \frac{\delta H}{\delta r}\delta r) \tag{A4}
$$

we have

$$
(A)_0 = 0 \tag{A5}
$$

$$
(Aq)_0 = \frac{\delta H}{\delta r} - q\frac{\delta H}{\delta v_2} = \frac{\delta H}{\delta \psi}, \tag{A6}
$$

$$
(A^* r)_0 = \frac{\delta H}{\delta q} - r\frac{\delta H}{\delta v_2} = -(\frac{\delta H}{\delta \phi})'. \tag{A7}
$$

Note that A, in fact, is a pure differential operator. Now from (A1) and (A2)

$$
\begin{aligned}
\frac{\delta H}{\delta K_{(1,2)}}\partial^{-1} &= (\partial^{-1}\frac{\delta H}{\delta v_1} + \frac{\delta H}{\delta v_2} + A)\partial^{-1} \\
&= (\partial^{-1}\frac{\delta H}{\delta v_2} + \partial^{-2}(\frac{\delta H}{\delta v_1} + (\frac{\delta H}{\delta v_2})') + A\partial^{-1}) + O(\partial^{-3}) \\
&= (\partial^{-1}\frac{\delta H}{\delta u_2} + \partial^{-2}\frac{\delta H}{\delta u_1} + A\partial^{-1}) + O(\partial^{-3}) \\
&= (\frac{\delta H}{\delta L_{(2,1)}})_- + A\partial^{-1} + O(\partial^{-3}).
\end{aligned}
\tag{A8}
$$

Let us define $B = A\partial^{-1}$, then

$$(B\phi)_0 = (A\partial^{-1}\phi)_0 = (Aq)_0 = \frac{\delta H}{\delta \psi}. \tag{A9}$$

On the other hand,

$$(B^*\psi)_0 = -(\partial^{-1}A^*r)_0 = -\int^x (A^*r)_0 = \frac{\delta H}{\delta \phi}. \tag{A10}$$

Eqs.(A9) and (A10) imply that

$$B = A\partial^{-1} = (\frac{\delta H}{\delta L_{(2,1)}})_+ \tag{A11}$$

and hence

$$\frac{\delta H}{\delta K_{(1,2)}}\partial^{-1} = \frac{\delta H}{\delta L_{(2,1)}} + O(\partial^{-3}). \tag{A12}$$

Combining (3.8) and (A12), it is easy to derive [18] the relation (3.13).

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